广义 Lotke-Volterra 生态模型的非线性奇摄动近似解

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摘要：非线性奇摄动问题在国际学术界中是一个重要的研究对象。它涉及到许多学科。在一些生态现象中，原始的研究方法只是采取某些简单观察和统计数据来得到结论。但是它对生态现象的实质的研究达不到效果。近来在国际上提出了研究生态学的动力学方法，即人们首先把它归化为代表它的现象本质的微分方程的模型，然后用数学方法来求解对应的方程，最后研究关于生物和数学理论的动力学方面的规律。目前，非线性摄动问题已经被广泛地研究。许多学者已经研究了一些近似理论。近似求解方法已被发展，包括平均法，边界层法，匹配渐近展开和多尺度法等等。研究非线性广义 Lotke-Volterra 捕食-被捕食生态模型，一个简单而有效的摄动方法被应用到捕食-被捕食生态模型。提出了捕食-被捕食的一个模型，它是一个微分方程系统，并用小的正参数按幂级数展开未知函数，然后得到关于幂级数的系数的方程，并求出它们的解。于是利用摄动方法得到了原问题解的渐近展开式。得到了它是原模型解是一个好的近似的结论，它是一个解析展开式并且能保持其解析运算。最后，给出了一个对应的例子，它说明得到的解具有很好的精度。

关键词：非线性；Lotke-Volterra 生态模型；奇摄动
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Nonlinear singularly perturbed approximate solution for generalized Lotke-Volterra ecological model

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Abstract: The nonlinear singularly perturbed problem is an important object of study in the international academic circles. It deals with many subjects. On the study of certain ecological phenomena, an original research adopts only some simple observational and statistical date to obtain the conclusion. But it cannot validly reflect its essence of the ecological phenomenon. Recently, the research some method of dynamics is produced for the study of ecology in international academic circles, i. e. the people first reduce it to the differential equation of model, which reflect its essential phenomenon and then solve the solution of the corresponding equation with mathematic methods; finally, study its dynamic rules upon the theory of biology and mathematics. Lately, the nonlinear perturbed problem has been widely investigated. Many scholars

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have considered the approximate theory. Approximate methods have been developed, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales and so on. This paper deal with the nonlinear generalized Lotke-Volterra prey-predator ecological model. A perturbation method, being simple and valid, is applied to study the prey-predator ecological model. The authors first provides a model of the prey-predator model, which is a system of differential equation and has developed the undermined functions in power series as small positive parameter. Then the equations of the coefficients for power series are obtained. And their solution is solved. Thus using the perturbation method the asymptotic expansions of solution for the original problem are obtained. The conclusion is that a good approximation for the original model comes to a solution, which is an analytic expression, and can keep on analytic operation. Lastly, a corresponding example is given, which show that obtained solution possesses a very good accuracy.

Key Words: nonlinear; Lotke-Volterra ecological model; singular perturbation

The nonlinear singularly perturbed problem is a very attractive object of study in the international academic circles [1]. Recently, approximate methods have been developed and refined, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales. Many scholars such as Ni and Wei [2], Marques [3], Bobkova [4] have done a great deal of works. Mo et al also considered a class of the shock layer solution of nonlinear equations for singularly perturbed problems [5], problems of atmospheric physics [6–8] and the ecological problems [9,10]. Recently, the research some method of dynamics is produced for the study of ecology in international academic circles [11,12]. The people first reduce it to the differential equation of model, which reflect its essential phenomenon and then solve the solution of the corresponding equation with mathematic methods. This paper deal with the nonlinear generalized Lotke-Volterra prey-predator ecological model. A perturbation method, being simple and valid, is applied to study the prey-predator ecological model. The authors first provides a model of the prey-predator model, which is a system of differential equation and has developed the undermined functions in power series as small positive parameter. And using a special and simple method, we studied a class of nonlinear singularly perturbed predator-prey ecological model.

1 Generalized Lotke-Volterra model

Now we consider the following nonlinear problem for the generalized Lotke-Volterra prey-predator model [3]:

$$\varepsilon \frac{\partial u_1}{\partial t} - a_1 u_1 + \gamma_1 u_1 u_2 = \varepsilon F_1(t, u_1, u_2), t \in [0, T] \quad (1)$$

$$\frac{\mathrm{d} u_2}{\mathrm{d} t} + a_2 u_2 - \gamma_2 u_1 u_2 = \varepsilon F_2(t, u_1, u_2), t \in [0, T] \quad (2)$$

$$u_1 = A_1(\varepsilon), t = 0 \quad (3)$$

$$u_2 = A_2(\varepsilon), t = 0 \quad (4)$$

where u_1 and u_2 stand for the quantities of prey and predator respectively, ε is a positive small parameter, a_i, γ_i and T are constants, a_1 represents the intrinsic growth rate of the prey, a_2 refers to the death rate of the predator, γ_i, $i = 1, 2$, are coefficients of the prey and predator respectively, the right hands εF_i, $i = 1, 2$, of the equations (1) and (2) are perturbation terms, F_i, A_i, $i = 1, 2$, are sufficiently smooth functions with regard to their variables in corresponding domains. We will construct the asymptotic expansions of the solution.

2 Outer solution

Let the expansions of a set of the outer solution (U_1, U_2) for the original problem (1)—(3) are

$$U_i - \sum_{j=0}^{\infty} U_{ij} \varepsilon^1, i = 1, 2 \quad (5)$$

Substituting (5) into (1), (2), (4), equating the coefficients of same powers of ε respectively, we obtain

$$a_1 U_{10} - \gamma_1 U_{10} U_{20} = 0 \tag{6}$$

$$\frac{\mathrm{d}U_{20}}{\mathrm{d}t} + a_2 U_{20} - \gamma_2 U_{10} U_{20} = 0 \tag{7}$$

$$U_{20} = A_2(0), t = 0 \tag{8}$$

$$(a_1 - \gamma_1 U_{20}) U_{1j} + \gamma_1 U_{10} U_{2j} = -F_{1j} \qquad j = 1, 2, \cdots \tag{9}$$

$$\frac{\mathrm{d}U_{2j}}{\mathrm{d}t} + (a_2 - \gamma_2 U_{10}) U_{2j} - \gamma_2 U_{20} U_{1j} = F_{2j} \qquad j = 1, 2, \cdots \tag{10}$$

$$U_{2j} = A_{2j} \qquad t = 0 \qquad j = 1, 2, \cdots \tag{11}$$

where $F_{ij}, A_{2j}, i = 1, 2, j = 1, 2, \cdots$, are determined functions successively, which constructions are omitted.

Obviously, from (6)—(8), we have

$$U_{10} = 0, U_{20} = A_2(0) \exp(-a_2 t) \tag{12}$$

We also have the solutions $(U_{1i}, U_{2i}), i = 1, 2, \cdots$, of the linear problems (9)—(11) successively:

$$U_{1j} = \frac{-F_{1j}}{a_1 - \gamma_1 A_2(0) \exp(-a_2 t)} \qquad j = 1, 2, \cdots \tag{13}$$

$$U_{2j} = \int_0^t \left[F_{2j} + \frac{\gamma_2 A_2(0) F_{1j}}{\gamma_1 A_2(0) - a_1 \exp(-a_2 t_1)}\right] \exp(-a_2(t - t_1)) dt_1 + A_{2j} \exp(-a_2 t), j = 1, 2, \cdots \tag{14}$$

Substituting (12)—(14) into (5), thus we obtain the outer solution (U_1, U_2) for the original nonlinear problem (1)—(4). But (U_1, U_2) may not satisfies condition (3), so we need to construct the initial layer correction (V_1, V_2).

3 Initial layer correction

Introducing a stretched variable [1] $\tau = \frac{t}{\varepsilon}$, let

$$u_i = U_i + V_i, i = 1, 2, \tag{15}$$

and

$$V_1 \sim \sum_{j=0}^{\infty} V_{ij} \varepsilon^j, i = 1, 2, \tag{16}$$

Substituting (15), (16) and (5) into (1)—(4), we obtain

$$\frac{\mathrm{d}V_{10}}{\mathrm{d}\tau} - (a_1 - \gamma_1 A_2(0)) V_{10} + \gamma_1 V_{10} V_{20} = 0 \tag{17}$$

$$\frac{\mathrm{d}V_{20}}{\mathrm{d}\tau} = 0 \tag{18}$$

$$V_{10}(0) = A_1(0), V_{20}(0) = 0 \tag{19}$$

$$\frac{\mathrm{d}V_{1j}}{\mathrm{d}\tau} - (a_1 - \gamma_1 A_2(0))) V_{1j} + \gamma_1 (V_{10} V_{2j} + V_{20} V_{1j}) = \bar{F}_{1,j} \qquad j = 1, 2, \cdots \tag{20}$$

$$\frac{\mathrm{d}V_{2j}}{\mathrm{d}\tau} = \bar{F}_{2j} \qquad j = 1, 2, \cdots \tag{21}$$

$$V_{1j}(0) = A_{1j} \qquad V_{2j}(0) = 0 \qquad j = 1, 2, \cdots \tag{22}$$

where $\bar{F}_{ij}, A_{1j}, i = 1, 2, j = 1, 2, \cdots$, are determined functions successively, which constructions are omitted too.

From (17)—(22), we have

$$V_{10} = A_1(0) \exp(a_1 - \gamma_1 A_2(0)) \tau, V_{20} = 0 \tag{23}$$

$$V_{1j} = \int_0^{\tau} \bar{G}_j(\tau_1) [\exp(a_1 - \gamma_1 A_2(0))(\tau - \tau_1)] \mathrm{d}\tau_1 + A_{1j} \exp(a_1 - \gamma_1 A_2(0)) \tau, V_{2j} = \int_0^{\tau} \bar{F}_{2j} \mathrm{d}\tau_1, j = 1, 2, \tag{24}$$

where $\bar{G}_j(\tau) = -\gamma_1 A_1(0)[\exp(a_1 - \gamma_1 A_2(0))\tau]\int_0^{\tau}\bar{F}_{2j}d\tau_1 + \bar{F}_{1j}, j = 1,2,\cdots$.

Then we obtain asymptotic expansions of solution for the problem (1)—(4):

$$u_i \sim \sum_{j=0}^{\infty}[U_{ij}(t), V_{ij}(\tau)]\varepsilon^i, t \in [0,T], i = 1,2,,$$

where $\varepsilon = \frac{t}{\varepsilon}$ and U_{ij}, V_{ij} are expressed by (2)—(4) and (13)—(14).

4 Example

We now consider a special nonlinear Volterra prey-predator perturbed model. The perturbed terms of the model (1)—(4) are $F_1 = F_2 = 0$, and $A_i(\varepsilon) = A_{i1} + A_{i2}\varepsilon$, where $A_{ij}, i-1,2, j = 0,1$ are constants. Thus we consider the following problem:

$$\varepsilon\frac{du_1}{dt} - a_1u_1 + \gamma_1u_1u_2 = 0, \frac{du_2}{dt} + a_2u_2 - \gamma_2u_1u_2 = 0, t \in (0,T] \quad (15)$$

$$u_1 = A_{10} + A_{11}\varepsilon, u_2 = A_{20} + A_{21}\varepsilon, t = 0 \quad (16)$$

From (2)—(4), we have

$$U_{1j} = 0, j = 0,1,2,\cdots,$$

$$U_{2j} = A_{2j}\exp(-a_2t), j = 0,1, U_{2j} = 0, j = 2,3,\cdots.$$

And from (13)—(14), we have

$$V_{1j} = A_{1j}exp(a_1 - \gamma_1A_{20})\tau, j = 0,1, V_{1j} = 0, j = 2,3,\cdots,$$

$$V_{2,j} = 0, j = 0,1,2,\cdots.$$

Then we obtain the asymptotic expansions of solution (u_1, u_2) for the Volterra prey-predator model and if $a_1 < \gamma_1A_{20}, a_2 > 0$, we can prove that the solution (u_1, u_2) of the nonlinear problem (15)—(16) is expressed by the following asymptotic expansions [4]:

$$u_1 = (A_{10} + A_{11}\varepsilon)\exp(a_1 - \gamma_2A)_{20})t/\varepsilon + O(\varepsilon^2), t \in [0,T], 0 < \varepsilon << 1,$$

$$u_2 = (A_{20} + A_{21}\varepsilon)\exp(-a_2t) + + O(\varepsilon^2), t \in [0,T], 0 < \varepsilon << 1.$$

In order to show the accuracy of above result, now we compare a special case. We give a set of parameters as follows: $a_i = A_i = 1, \gamma_i = 2, i = 1,2, \varepsilon = 0.01$. We establish a comparison of the values of between (u_{1num}, u_{2num}) for the numerical solution and (u_{1asy}, u_{2asy}) of the asymptotic solution of the perturbation method. We obtain the following results (see the Fig. 1, Fig. 2, Table 1 and Table 2).

Table 1 A comparison of the values of between u_{1nmu} and u_{1asy}

t	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.10
u_{1num}	1.010	0.676	0.248	0.094	0.036	0.013	0.005	0.002	0.001	0.000	0.000
u_{1asy}	1.010	0.699	0.239	0.090	0.034	0.011	0.004	0.002	0.001	0.000	0.000

Table 2 A comparison of the values of between u_{2nmu} and u_{2asy}

t	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
u_{2num}	1.01	1.01	0.99	0.97	0.95	0.92	0.88	0.78	0.61	0.48	0.38
u_{2asy}	1.01	0.99	0.97	0.95	0.93	0.90	0.86	0.76	0.60	0.47	0.37

From the result of the Figs. and Tables, we know that the values of between the solution $(u_{1nmu}u_{2nmu})$, of the numerical calculation and the (u_{1asy}, u_{2asy}) of the asymptotic method are small. Thus using the perturbation method, there is a good accuracy for the computed values of the model (1)—(4).

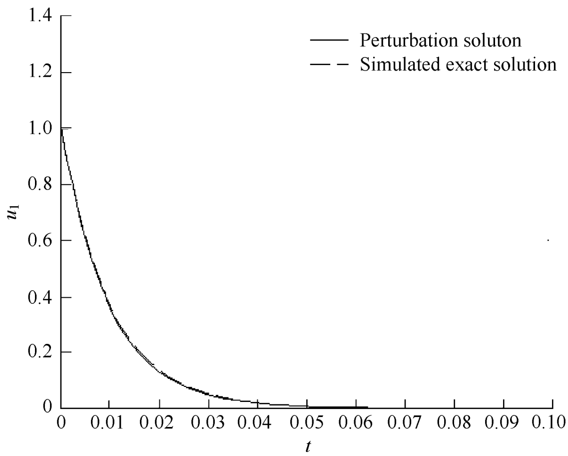

Fig. 1 Comparison between simulation solution $u_{1\mathrm{mun}}$ and perturbed asymptotic solution $u_{1\mathrm{asy}}$

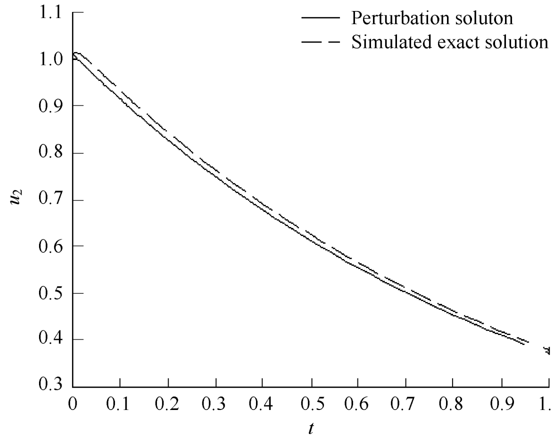

Fig. 2 Comparison between simulation solution $u_{2\mathrm{mun}}$ and perturbed asymptotic solution $u_{2\mathrm{asy}}$

5 Conclusions

() The prey-predator ecological models are a complicated natural phenomenon. Hence we need to reduce basic models and solve them by using the approximate method. The singularly perturbed solving method is a simple and valid method.

(2) The singularly perturbed solving method is an approximate method, which differs from general numerical method. The expansions of solution through the singularly perturbed solving method can be kept in the analytic operation. Thus, we can further study that the fixed quality and quantitative behaviors of the solution u_i, $i = 1, 2$ for the Volterra predator-prey ecological model. But we do not discuss any more.

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