第 26 卷第 1 期
2006 年 1 月

生 态 学 报
ACTA ECOLOGICA SINICA

Vol.26, No.1
Jan., 2006

HIV 传播的人群生态动力学模型

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摘要：研究了 HIV 传播的动力学模型，描述了流行性传染病区域的人群传播规律，特别是利用摄动理论对艾滋病的传播动力学非线性方程作了定量、定性的讨论。

关键词：HIV 传播；艾滋病；非线性；摄动

文章编号：1000-0933(2006)01-0104-04 **中图分类号**：Q141 **文献标识码**：A

Bionomics dynamic model of human groups for HIV transmission

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Abstract: The studies on the epidemic contagion transmission are of high value and have received an adequate attention at all times. Especially, the transmission of HIV virus has attached more importance to scientists. There is grievous calamity to human. It brings severe menace. On the study of the transmission for HIV virus, an original research adopts only some simple observational and statistical data to obtain the conclusion. But it can not validly reflect its essence of the transmission. Recently, the research method of dynamics is produced for the study of HIV's transmission in international academic circles, i. e. the people first reduce it to the differential equation of model, which reflect its essential phenomenon and then solve the solution of the corresponding equation with the mathematic methods; finally, study its dynamic rules upon the theory of biology, medicine and mathematics. This paper deals with the study of the HIV's transmission for a corresponding nonlinear dynamic model by using the modern mathematic perturbation theory. Lately, the nonlinear perturbed problem has been widely investigated in the international academic circles. Many scholars have considered the approximate theory. Approximate methods have been developed and refined, including the average method, boundary layer method, matched asymptotic expansion method and multiple scales method. In this paper, a perturbed method, being simple and valid, is applied to study the epidemic contagion transmission. The author first establishes a model of the epidemic contagion transmission, which is a system of differential equation, and has developed the undetermined functions in power series as small positive parameter. Then the equations of the coefficients for power series are obtained. Their solutions are solved. Thus, the conclusion is that a good approximate for the original model comes to a solution, which is an analytic expression, and can keep on analytic operation.

Key words: HIV transmission; AIDS; nonlinear; perturbation

流行性传染病的传播，一直是医学界、生态环境学界十分关注的对象。特别是艾滋病这类人体免疫缺损病毒(HIV)的传播，更值得人们重视。它对人类健康带来严重的威胁。对于 HIV 传播的研究，最初只是局限

基金项目：国家自然科学基金资助项目(90111011，10471039)；国家重点基础研究发展计划资助项目(2003CB415101-03；2004CB418304)；中国科学院创新方向性资助项目(KZCX3-SW-221)；上海市教育委员会 E-研究院建设计划资助项目(N.E03004)

收稿日期：2005-06-28；**修订日期**：2005-11-15

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Foundation item: the National Natural Science Foundation of China (No. 90111011 and 10471039), the National Key Project for Basics Research (No. 2003CB415101-03 and 2004CB418304), the Key Project of Chinese Academy of Sciences (No. KZCX3-SW-221) and in part by E-Insitutes of Shanghai Municipal Education Commission (N. E03004)

Received date: 2005-06-28; **Accepted date**: 2005-11-15

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于采用一些简单的观察数据和简单的统计数据来作出分析和推断。这不能有效的反映其传播的本质。目前关于艾滋病的传播研究,在国际学术界已经将它归纳为动力学的研究方法[1-3]。即将它归化为一个反映其基本现象的微分方程模型。然后利用数学的解析方法求出对应方程的解。最后将求得的结果作为依据,从生物、医学、数学交叉学科理论,来研究其动态规律。本文就是以相应的一个非线性动力学模型为基础,利用现代数学中的摄动理论工具,从人群生态学的观点来研究 HIV 的传播。

非线性问题在国际学术界是一个十分关注的对象。近来,许多学者做了大量的工作[4-6]。莫嘉琪等也研究了非线性奇摄动问题[7-14]。本文是利用一个简单而有效的摄动方法来研究 HIV 传染人群中的一类非线性模型。

1　HIV 人群传播动力学模型

今考虑如下一个 HIV 传播人群的生态动力学微分系统的模型[1,3]:

$$\frac{dx}{dt} = \alpha xy - \beta x \tag{1}$$

$$\frac{dy}{dt} = -\alpha xy - \gamma x^2 y + \delta x + c \tag{2}$$

式中,$x(t)$表示在 HIV 传播区域内的感染者人数,$y(t)$为易感者人数,t 为时间,$c \geqslant 0$ 为易感者的出生率,$\alpha,\beta,\gamma,\delta$ 为常数。在系统(1),(2)式中,αxy 项表示感染者与易感者因"交感"而造成的患者的增加速度,$-\beta x$项表示由于患者死亡而引起的患者的减少速度,$-\alpha xy$项表示感染者与易感者"交感"易感者变为患者后使得易感者减少的速度,$-\gamma x^2 y$ 项表示采取防疫措施后使得易感者减少的速度,δx 项表示患者增多时易感者的增加率。系统(1),(2)是一个典型的在患区人群的 HIV 传播的生态动力学模型。将构造模型(1),(2)解的渐近表达式,并从而可以将所得的表示式来研究 HIV 的传播性态和规律。在本文中,鉴于突出本文的主要研究重点,仅考虑当参数为正的相对小量的情形。这将使研究的"传播区域"的范围更为广泛。

2　摄动解

系统(1),(2)式的未知函数 x,y 实际上是参数 α 的"函数"。所以不妨分别以 $x_\alpha(t),y_\alpha(t)$记之,并且将它们分别写为关于 α 的幂级数:

$$x_\alpha(t) = \sum_{i=0}^{\infty} x_i(t)\alpha^i,\ y_\alpha(t) = \sum_{i=0}^{\infty} y_i(t)\alpha^i \tag{3}$$

将(3)式代入(1),(2)式,按 α 展开非线性项,合并 α 的同次幂项,并分别令各次幂的系数为零。于是,由 α 的零次幂的系数可得:

$$\frac{dx_0}{dt} + \beta x_0 = 0 \tag{4}$$

$$\frac{dy_0}{dt} + \gamma x_0^2 y_0 = \delta x_0 + c \tag{5}$$

由 $\alpha^j,j=1,2,\cdots$,的系数为零,可得:

$$\frac{dx_j}{dt} + \beta x_j = \sum_{k=0}^{j-1} x_k y_{y-k-1} \tag{6}$$

$$\frac{dy_j}{dt} - \delta x_j = \sum_{k=0}^{j-1} x_k y_{y-k-1} - \gamma \sum_{k=0}^{j} \Big[\sum_{i=0}^{k} x_i x_{k-i}\Big] y_{j-k} \tag{7}$$

由(4),(5)式,不难得到

$$x_0(t) = C_0 \exp(-\beta t) \tag{8}$$

$$y_0(t) = D_0 \exp\left(\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + Q_1(t) \tag{9}$$

其中:

$$Q_1(t) = \exp\left(\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right)\int_0^t[\delta C_0\exp(-\beta t_1) + c]\exp\left(-\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t_1)\right)dt_1 \tag{10}$$

而 C_0 和 D_0 为任意常数,它们可由系统的初始状态来决定。

在(6),(7)式中取 $j=1$ 的情形,并考虑到(8),(9)式,有:

$$\frac{dx_1}{dt} + \beta x_1 = C_0 D_0 \exp\left(-\beta t + \frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + C_0 Q_1(t)\exp(-\beta t) \tag{11}$$

$$\frac{dy_1}{dt} + \gamma C_0^2[\exp(-2\beta t)]y_1 = -C_0 D_0\exp\left(-\beta t + \frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + C_0 Q_1(t)\exp(-\beta t) - \gamma[C_0^2\exp(-2\beta t) + 2C_0(\exp(-\beta t))x_1]\left[D_0\exp\left(\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + Q_1(t)\right] + \delta x_1 \tag{12}$$

由(11)式,可以求得:

$$x_1(t) = C_1\exp(-\beta t) + \int_0^t Q_2(t_1)\exp(\beta t_1)dt_1 \tag{13}$$

其中:

$$Q_2(t) = C_0 D_0\exp\left(-\beta t + \frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + C_0 Q_1(t)\exp(-\beta t) \tag{14}$$

再将(13)式代入(12)式:

$$\frac{dy_1}{dt} + \gamma C_0^2[\exp(-2\beta t)]y_1 = Q_3(t) \tag{15}$$

其中:

$$Q_3(t) = -C_0 D_0\exp\left(-\beta t + \frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + C_0 Q_1(t)\exp(-\beta t) - \gamma[C_0^2\exp(-2\beta t) + 2(C_0\exp(-\beta t))x_1]\left[D_0\exp\left(\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + Q_1(t)\right] + \delta x_1 \tag{16}$$

式中的 x_1 由(13)式表示。

由线性方程(15)可以得到解:

$$y_1(t) = D_1\exp\left[\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right] + \exp\left[\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right]\int_0^t Q_3(t_1)\exp\left[-\frac{C_0^2}{2\beta}\exp(-2\beta t_1)\right]dt_1 \tag{17}$$

其中(13)式和(17)式中的 C_1 和 D_1 为任意常数,它们也可由系统的初始状态来决定。

由所得的结果(8),(9),(13),(17)式,便得到模型系统(1),(2)式解的一次近似的渐近展开式

$$x_\alpha(t) = C_0\exp(-\beta t) + \left[C_1\exp(-\beta t) + \int_0^t Q_2(t_1)\exp)(\beta t_1)dt_1\right]\alpha + O(\alpha^2)$$

$$0 < \alpha \ll 1$$

$$y_\alpha(t) = D_0\exp\left(\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right) + Q_1(t) + \left[D_1\exp\left[\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right] + \exp\left[\frac{\gamma C_0^2}{2\beta}\exp(-2\beta t)\right]\int_0^t Q_3(t_1)\exp\left[-\frac{C_0^2}{2\beta}\exp(-2\beta t_1)\right]dt_1\right]\alpha + O(\alpha^2)$$

$$0 < \alpha \ll 1$$

式中 Q_i, $i=1,2,3$ 分别由(10),(14),(16)式表示。

利用类似的方法,由系统(6),(7)式,能得到系统(1),(2)式的更高次的渐近解。

3 结语

(1) 由问题(1)式,(2)式的结构,能够证明由上述方法得到上述解的渐近展开式是关于 α 为一致有效的。

(2) HIV 传播从生态学的观点来说,是一个相当复杂的现象。把它归化为数学上的动力学问题,然后用

数学非线性模型处理方法去得到足够精度的近似解。这是研究HIV人群的传播问题的一个有效的途径。本文中采用的摄动理论就是一个简单而有效的方法。

(3) 从数学理论观点来看,摄动方法是一个解析的方法。它不同于一般的数值求解方法,更不是简单的模拟方法。本方法得到的解的表示式,还能继续进行解析运算。事实上,可以由得到的渐近展开式进一步进行定性和定量的研究。例如,进一步用微分的方法算出感染者和易感染者数量的变化速度、画出上述两者在不同时间的数量变化曲线,从而发现其规律,并预报感染者和易感染者在一定时期内的数量、发展趋向的规律等等。然而,利用数值解法或单纯地模拟就难以更深入、更精确的讨论。

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